1. 设 $f: I\to\mathbb{R}$ 为凸函数，$I=[-2,2]$，设 $f_r'(-2)=a$，$f_l'(2)=b$

(1) 求证 f 在 $(-2,2)$ 上至多可数个点不可导.

Pf:

If $x_1<x<x_2$, then $x=\lambda_1 x_1+\lambda_2 x_2$ for $\lambda_1=\dfrac{x_2-x}{x_2-x_1}$, $\lambda_2=(1-\lambda_1)=\dfrac{x-x_1}{x_2-x_1}$, then

$$(x_2-x_1)f(x)\le(x_2-x)f(x_1)+(x-x_1)f(x_2)$$

$$\Rightarrow \frac{f(x)-f(x_1)}{x-x_1}\le\frac{f(x_2)-f(x)}{x_2-x}\Rightarrow\frac{f(x)-f(x_1)}{x-x_1}\le\frac{f(x_2)-f(x_1)}{x_2-x_1}\le\frac{f(x_2)-f(x)}{x_2-x}.$$

Hence we have $\dfrac{f(x+h)-f(x)}{h}$ is an increasing function of h when $x+h\in I$ and $h\neq 0$.

$\Rightarrow \forall x\in I$, $f_l'(x)$ and $f_r'(x)$ exist. If $-2<x_1<x_2<2$, then

$$a=f_r'(-2)\le\frac{f(x_1)-f(-2)}{x_1-(-2)}\le f_l'(x_1)\le f_r'(x_1)\le\frac{(f(x_2)-f(x_1))}{x_2-x_1}\le f_l'(x_2)\le f_r'(x_2)\le\frac{f(2)-f(x_2)}{2-x_2}\le f_l'(2)=b.$$

Now we define

$$X=\{x\in I:\ f\text{ is not differentiable at }x\text{ i.e. }f_r'(x)>f_l'(x)\}.$$

Suppose the X is uncountable, we define $X_0=\{x\in X:\ f_r'(x)-f_l'(x)>1\}$ and $X_n=\{x\in X:\ 2^{-n}<f_r'(x)-f_l'(x)\le 2^{-n+1}\}$, $\forall n\ge 1$, then $\exists\, n_0\in\mathbb{N}$ s.t. X_{n_0} is countable. It implies that

$$\sum_{x\in X_{n_0}} f_r'(x)-f_l'(x)\ge\sum_{x\in X_{n_0}}2^{-n_0}=+\infty.$$

Which contradicts to
$$\sum_{x\in X_{n_0}} f_r'(x)-f_l'(x)\le f_l'(2)-f_r'(-2)=b-a<+\infty$$
□

(2). 令 $f^*(x)=\sup\limits_{y\in[-2,2]}(xy-f(y))$，$\forall x\in[a,b]$. 求证：$f^*:[a,b]\to\mathbb{R}$ 为凸函数.

Pf:

$\forall x_1,x_2\in[a,b]$, $\lambda_1,\lambda_2\ge 0$, $\lambda_1+\lambda_2=1$, we have

$$f^*(\lambda_1x_1+\lambda_2x_2)=\sup_{y\in[-2,2]}\left((\lambda_1x_1+\lambda_2x_2)y-f(y)\right)$$

$$=\sup\ \left(\lambda_1(x_1y-f(y))+\lambda_2(x_2y-f(y))\right)$$

$$= \sup_{y\in[-2,2]} \left(\lambda_1 (x_1 \cdot y - f(y)) + \lambda_2 (x_2 \cdot y - f(y)) \right)$$

$$\leq \sup_{y\in[-2,2]} \lambda_1 \cdot (x_1 \cdot y - f(y)) + \sup_{y\in[-2,2]} \lambda_2 \cdot (x_2 \cdot y - f(y))$$

$$= \lambda_1 f^*(x_1) + \lambda_2 f^*(x_2)$$

(3). 求证：$(f^*)^* = f$.

Pf: From the definition we obtain at once

$$f^*(y) + f(x) \geq x \cdot y, \quad \forall x \in [-2,2],\ y \in [a,b].$$

$$\Rightarrow \quad f(x) \geq x \cdot y - f^*(y), \quad \forall x \in [-2,2],\ y \in [a,b].$$

$$\Rightarrow \quad f(x) \geq (f^*)^*(x) \quad \forall x \in [-2,2].$$

$\forall x \in I$, $f_l'(x) \leq f_r'(x)$, we can choose $y \in [a,b]$ s.t. $f_l'(x) \leq y \leq f_r'(x)$, then

$$\frac{f(x) - f(z_1)}{x - z_1} \leq f_l'(x) \leq y \leq f_r'(x) \leq \frac{f(z_2) - f(x)}{z_2 - x}, \quad \forall z_2 > x > z_1.$$

$$\Rightarrow \quad yz - f(z) \leq y \cdot x - f(x), \quad \forall z \in I$$

$$\Rightarrow \quad f^*(y) \leq y \cdot x - f(x) \quad \text{for some } y \in [a,b]$$

$$\Rightarrow \quad f(x) \leq y \cdot x - f^*(y) \leq (f^*)^*(x)$$

□

(4). 若 f 在 $[-2,2]$ 上可导，求证：对于 $y = f'(x) \in [a,b]$，有 $f^*(y) = x \cdot f'(x) - f(x)$.

Pf:

$$f^*(f'(x)) = \sup_{y\in[-2,2]} \{ f'(x) \cdot y - f(y) \}, \quad \text{let } g(y) = f'(x) \cdot y - f(y),\ y \in [-2,2]$$

Then $g'(y) = f'(x) - f'(y)$, since $f'(y)$ is nondecreasing, we get

$$g'(y) \leq 0 \text{ on } [-2,x], \quad g'(y) \geq 0 \text{ on } [x,2].$$

$$\Rightarrow \quad \sup_{y\in[-2,2]} g(y) = g(x) = f'(x) \cdot x - f(x)$$

□

(5). 若 $f\in C^2([-2,2])$, $f^*\in C^2([a,b])$, 则 $\forall x\in(-2,2)$, 有 $f''(x)\cdot(f^*)''(f'(x))=1$.

Pf:

$(f^*)'(f'(x)) = x$

We only consider the case where $a<b$.

Claim 1: $f'(x)$ is strictly increasing on $[-2,2]$.

Pf:

Otherwise, $\exists\, x_0<y_0$ s.t. $f'(x_0)<f'(y_0)$. WLOG, we may assume that $f'(y_0)<b$ and we replace y_0 by

$$y_0 := \sup\{x\in I : f'(x)=f'(x_0)\} < 2.$$

Thus $f'(x)\equiv f'(x_0)$ on $[x_0,y_0]$ and $f'(x)>f'(x_0)$ on $(y_0,2)$.

$\forall x\in(y_0,2)$, $f'(x)>f'(x_0) \xRightarrow{\text{Lagrange}} \exists\{x_n\}\subset(y_0,2)$ s.t. $x_n\to y_0$ and $f''(x_n)>0$.

From Ex(4), $(f^*)(f'(x)) = x\cdot f'(x)-f(x)$, $\forall x\in[-2,2]$, then

$$(f^*)'(f'(x))\cdot f''(x) = x\cdot f''(x).$$

$$\Rightarrow\quad (f^*)'(f'(x_n))\cdot f''(x_n) = x_n\cdot f''(x_n) \xRightarrow{f''(x_n)>0} (f^*)'(f'(x_n)) = x_n$$

Since $f, f^*\in C^2$, let $n\to\infty$, $(f^*)'(f'(y_0)) = y_0$.

We have

$$\begin{aligned}
&(f^*)''(f'(y_0))\\
&= \lim_{n\to\infty}\frac{(f^*)'(f'(x_n))-(f^*)'(f'(y_0))}{f'(x_n)-f'(y_0)} \qquad \left(\begin{array}{l} f'(x_n)\to f'(y_0)\\ \&\ f'(x_n)>f'(y_0)\end{array}\right)\\
&= \lim_{n\to\infty}\frac{x_n-y_0}{f'(x_n)-f'(y_0)}\\
&= \frac{1}{f''(y_0)}.
\end{aligned}$$

On the other hand, $f'(x)\equiv f'(y_0)$, $\forall x\in[x_0,y_0] \Rightarrow f''(y_0)=0$, which contradicts $f''(y_0)=\dfrac{1}{(f^*)''(f'(y_0))}\neq 0$. □

Claim 2: $f''(x)>0$ on $[-2,2]$.

Pf:

Suppose not, $\exists\, x_0$ s.t. $f''(x_0)=0$.

Since f is strictly increasing, then $\forall n \in \mathbb{N}$, $\exists x_n \in (x_0 - \frac{1}{n}, x_0 + \frac{1}{n})$ s.t. $f''(x_n) > 0$. Otherwise, $f''(x) \equiv 0$ on $(x_0 - \frac{1}{n}, x_0 + \frac{1}{n})$ for some n which implies that $f'(x) \equiv$ constant on $(x_0 - \frac{1}{n}, x_0 + \frac{1}{n})$.

Let's repeat what we did in the proof of Claim 1, we have

$$(f^*)''(f'(x_0)) = \frac{1}{f''(x_0)}$$

which contradicts to $f''(x_0) = 0$ $\square$

Proof of (5):

From (4), $(f^*)(f'(x)) = x \cdot f'(x) - f(x)$, $\forall x \in [-2,2]$, then

$$(f^*)'(f'(x)) \cdot f''(x) = x \cdot f''(x),$$

$$\xRightarrow{f''(x) > 0} \quad (f^*)'(f'(x)) = x$$

$$\Longrightarrow \quad (f^*)''(f'(x)) \cdot f''(x) = 1, \ \forall x \in [-2,2]. \qquad \square$$